Building an electromagnetic pendulum to have true isochronism

Abstract

Galileo found isochronism of mechanical pendulum but failed to demonstrate it[1]. In this internal assessment, the period of a traditional pendulum is proved dependent on the swing angle by a novel simulation. A new pendulum switching electromagnetic energies is built to have true isochronism. The design philosophy and simulation are presented. An experiment will be carried out to validate this electrical pendulum.

Background Information and Motivation

A pendulum is a device that makes transfer in cycles of **gravitational potential energy and potential energy**[2]. Its periodicity provides a reliable way to count time. The concept of pendulum traces back to Galileo noticing the regularity of a suspended lamp swinging back and forth in the Duomo cathedral in Pisa. He declared that the oscillation period of length-fixed pendulums is constant, i.e. isochronous, regardless of the amplitude of the oscillation[1]. Although this assertion has been well received, it conflicts with my experience in playing on the swing that larger swings take a longer time to return.

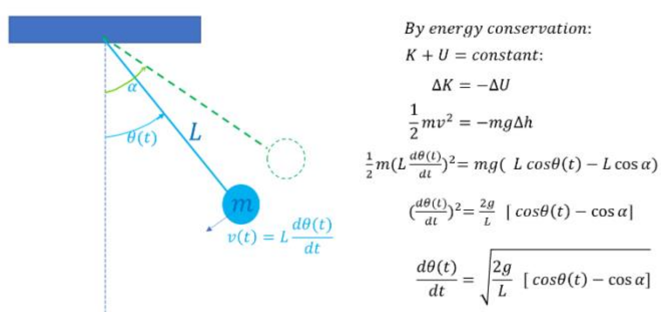


Figure 1

The survey on the internet proves I'm not too skeptical. *'From 1602 onwards, Galileo was referred to pendulum isochronism as an admirable property but failed to demonstrate it.[1]'* Therefore, I decide to help with Galileo moving on demonstrating isochronism with technology nowadays. By energy conversion, I derived the differential equation describing the time function of angle first. $\frac{d\theta(t)}{dt} = \sqrt{\frac{2g}{L}}[cos\theta(t) - cos\alpha], where\ \alpha = intial\ swing\ angle$. *(Figure 1)*

I adopt a numerical method developed in my math internal assessment to solve the nonlinear differential equation[2].

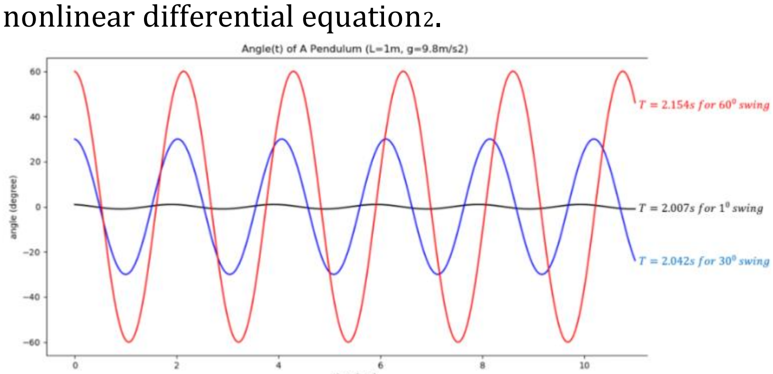


Figure 2

It shows the period varies as its initial swing to swing. (*Figure 2)*

My simulation results come in accordance with the formula derived from the university of Conneticut., as cited below3. *(Figure 3)*

$$Period : T = T_0 \left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6 + \cdots \right] \quad where\ k = \sin\frac{\alpha}{2},\ \ T_0 = 2\pi\sqrt{\frac{L}{g}}$$

Figure 3

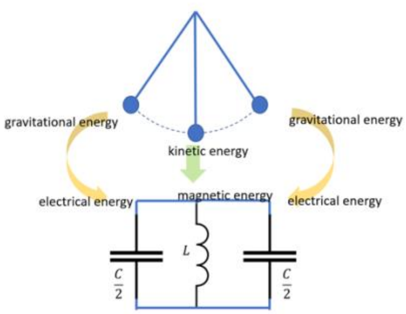


Figure 4

Since the mechanical pendulum fails the isochronism in strict sense, I was motivated to explore alternative mechanism to make good periodicity.

Research

Looking into the swing of a pendulum, it transfers energy back and forth in **gravitational-kinetic-gravitational** conversion. If I have energy storage devices similarly connected in **electrical-magnetic-electrical topology**, the energy might get swinging accordingly.

As we know, electrical energy could be stored in the electrical field of a capacitor while an inductor builds a magnetic field around it to keep magnetic energy. It's intuitive, with comparison to a mechanical pendulum, to make the electrical pendulum on the left. (*Figure 4*)

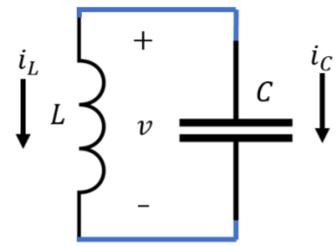
Research: Theory of Operation

A capacitor is well introduced in physics by $q_c(t) = Cv_c(t)$, where C is the capacitance and q(t) are the charge and voltage across the capacitance. By differentiating the equation, we get capacitor current $i(t) = \frac{dq_c(t)}{dt} = \frac{d(Cv_c(t))}{dt} = C\frac{dv_c(t)}{dt}$.

The formula counterpart for an inductor is never introduced in high school physics even though it's the object Faraday's and Ampere's laws mainly address. Faraday's law says the electromotive force, (emf, $\varepsilon(t)$), equals to the change rate of magnetic flux, $\varepsilon(t) = -\frac{d[B(t)A(t)]}{dt}$, where B(t) is the magnetic field and A(t) is the area enclosed by a loop. It's simplified to be $\varepsilon(t) = -A\frac{dB(t)}{dt}$ for an inductor of which the area is constant. By Ampere's law that states magnetic field is proportionally generated by current flow, we know for an inductor, $B(t) = k_{BI} \cdot i_L(t)$, where $i_L(t)$ is inductor current and k_{BI} is just a conversion constant dependent on inductor structure and material around. The emf across an inductor becomes $\varepsilon(t) = -A\frac{dB(t)}{dt} = -A\frac{d(k_{BI}\cdot i_L(t))}{dt} = -k_{BI}A\frac{d_{i_L}(t)}{dt}$. One-coulomb charges will lose energy 'by potential difference, $V_L(t)$' while gaining energy by emf when passing through an inductor. According to energy conversion, $v_L(t) + \varepsilon(t) = 0$, we get $v_L(t) = -\varepsilon(t) = k_{BI}A\frac{di_L(t)}{dt} = L\frac{di_L(t)}{dt}$ where L is called the inductance of the inductor.

Now that we have got the *I-V* characteristics of capacitor $\left(i_c(t) = C\frac{dv_C(t)}{dt}\right)$ and inductor $(v_L(t) = L\frac{di_L(t)}{dt})$, we are ready to further analyze the *C-L-C* topology.

Research Question



Through my research, I'm intrigued to find out how possible is it to create a pendulum that is able to achieve isochronism. This has helped me to draw up my research question, *'Can I build a pendulum of true isochronism?'*

Question(A): Can the electrical pendulum really swing?

Two parallel identical capacitors could be combined as one of double capacitance. The *C-L-C* circuitry is equivalent a new L-C one. (*Figure 5*)

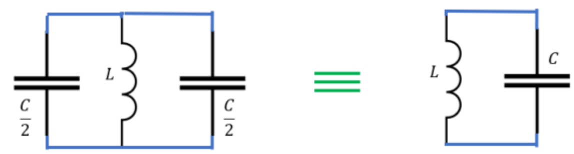


Figure 5

The time function of voltage could be derived from below with individual device I-V characteristic and Kirchhoff current law applied:

Capacitor: $i_c(t) = C\frac{dv_C(t)}{dt} = C\frac{dv(t)}{dt}$

Inductor: $v_L(t) = v_L(t) = L\frac{di_L(t)}{dt}$

Kirchhoff's current law, $i_C(t) + i_L(t) = 0$

$$v(t) = L\frac{di_L(t)}{dt} = -L\frac{d_{i_c}(t)}{dt} - L\frac{d\left(C\frac{dv(t)}{dt}\right)}{dt} = -LC\frac{d^2v(t)}{dt^2}$$

$$\rightarrow \frac{d^2v}{dt^2} = -\frac{1}{LC}v(t)$$

Though a neat equation is derived, it seems impossible for me to solve the equation. Luckily, I can recall the *S.H.M* solution, $x(t) = Asin2\pi\frac{1}{T_0}t\ where\ T_0 = 2\pi\sqrt{\frac{m}{k}}$ for differential equation, $\frac{d^2x(t)}{dt^2} = -\frac{k}{m}x(t)$. By comparison, it can be inferred that $\frac{d^2v(t)}{dt^2} = -\frac{1}{LC}v(t)$ must have its solution in similar form, $v(t) = Asin2\pi\frac{1}{T_0}t, where\ T_0 = 2\pi\sqrt{LC}$.

It mathematically proves that the voltage of *L-C* circuitry is a periodic function of period $T_0 = 2\pi\sqrt{LC}$. Unlike a pendulum to vary its period with swing angle, this electrical pendulum has period independent of oscillation amplitude, suggesting it truly makes isochronism.

I also simulated the voltage waveform by *Partsim*, an online circuit simulator. (*Figure 6*)

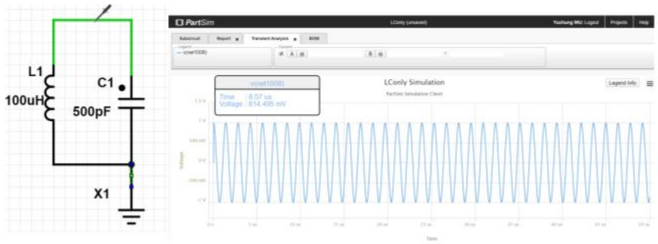
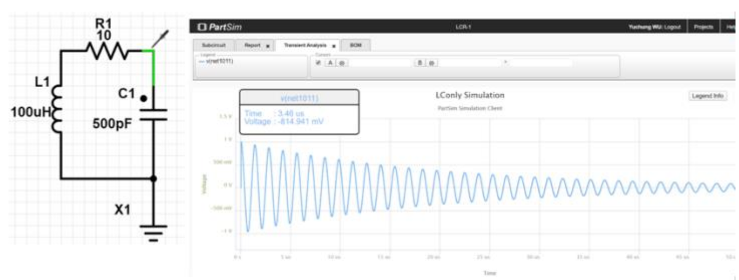


Figure 6

It alternatively confirms that the voltage of *L-C* circuitry is truly sine wave as I have mathematically inferred.

Research Question(B): Can the electrical pendulum swing for long?

Even though we show oscillations could happen, it doesn't sufficiently mean it will carry on forever. Notice that $v(t) = 0$ is a trivial solution to the mathematical equation. A pendulum swing dies away due to friction if left unattended. Similarly, the "electrical pendulum's swing" will stop because of the inevitable existence of electrical friction, resistance. I have added a resistor to reflect the inevitable resistance and the simulation result confirms my concern.



Trying to keep the swing last longer seems too passive. Could we aggressively 'grow' a tiny swing?

For a mechanical pendulum, a large swing could build from a tiny one if we 'tap' it at the right time to injecting positive work, applying forces in the same direction of pendulum movement. Could we tap our 'electromagnetic pendulum' by injecting energy similarly?

To inject the energy, we need a controllable device to pass energy.

This device needs to serve 2 functions:

1. A source path under control to supply *L-C* circuitry the energy
2. A sensor detecting the dynamic change inside *L-C* circuitry for us to control the source path

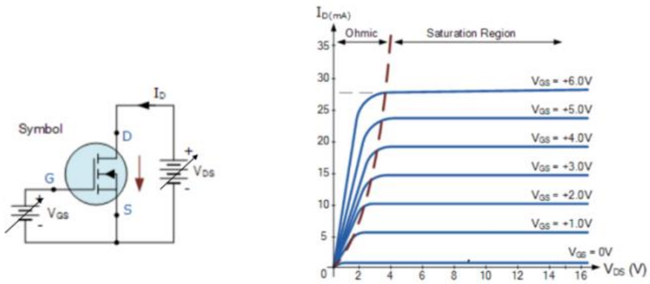
A *MOSFET*, a well-known controllable device, could probably serve the function.

Implementation

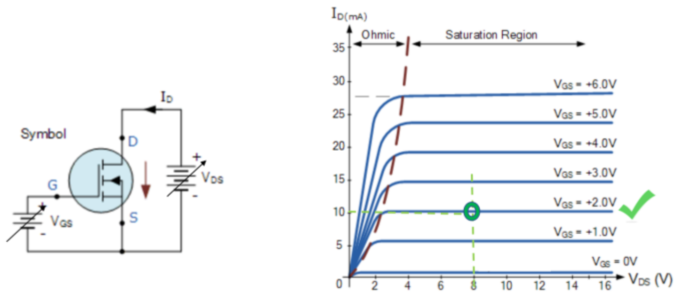
Implementation 1: Understand a MOSFET

A *MOSFET* (Metal-Oxide-Semiconductor Field-Effector Transistor) is a 3-terminal (Gate, Drain, Source) device. College textbooks spend chapters to introduce them[4]. It's not necessary to get into much details for now and the *I-V* characterize should fully serve me.

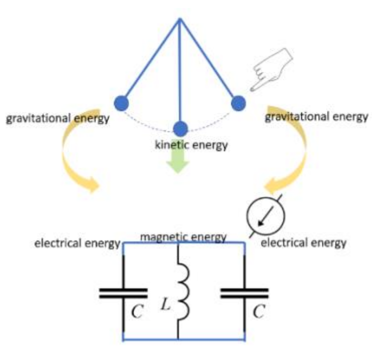
The current I_D flowing through terminal *D* and *S* is dependent on V_{GS} and V_{DS}.

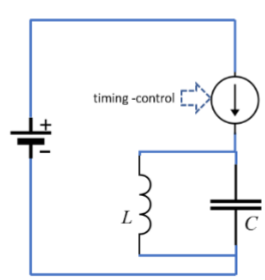


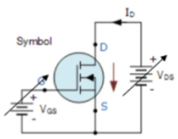
For example, we can learn I_D*=10mA* when V_{GS} *= 2V* and V_{DS}*= 8V* by looking up the figure.

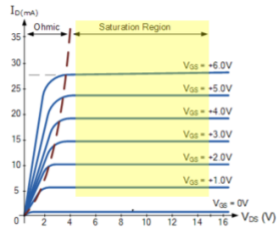


For large V_{DS}, the current of I_D flowing through terminal *D* and *S* depends on V_{GS} only (the yellow region). It could be easily told because the I_D traces are flat inside the yellow region. Equivalently, in this special condition named saturation region, I_D is controlled only by V_{GS}.

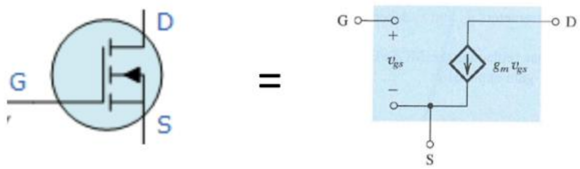








Then in a saturation state, the *MOSFET* could be simplified as a current source. The current is controlled by V_{GS} with proportional coefficient g_m which is positive but is not necessarily a constant.

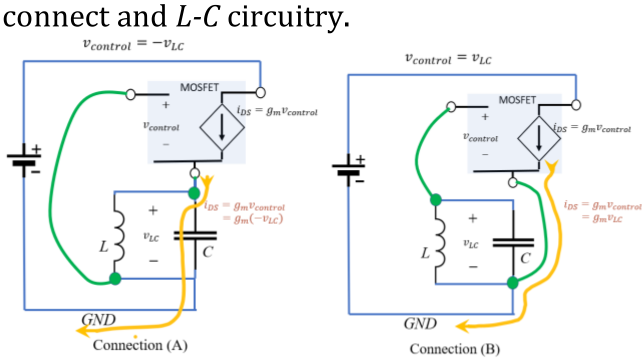


Next, let us replace the dream device with a practical *MOSFET*.

Implementation 2: Building the circuit

A *MOSFET* seems to provide similar functions to the dream device. However, the dream device has the sensor and current source separated while a *MOSFET* has its sensor and the source share a common terminal. I'll practically replace the dream device with a real *MOSFET* to see whether it causes a problem.

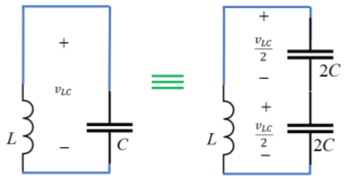
It's obvious that we can have only 2 possibilities to connect the sensing part to connect and *L-C* circuitry.



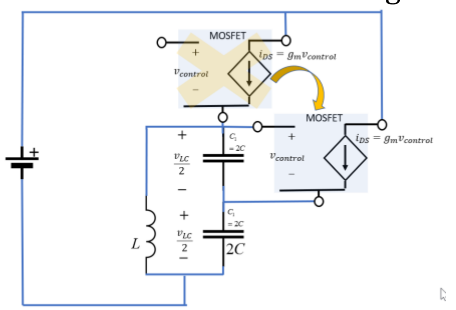
Connection A: The power into LC is $i_{DS} \times v_{LC} = g_m v_{control} \times (-v_{control}) = -g_m {v_{control}}^2 < 0$. It takes energy out of the *LC* tank.

Connection B: The current i_{DS} doesn't even flow into *L* or *C*. The power injected into *LC* circuitry is zero.

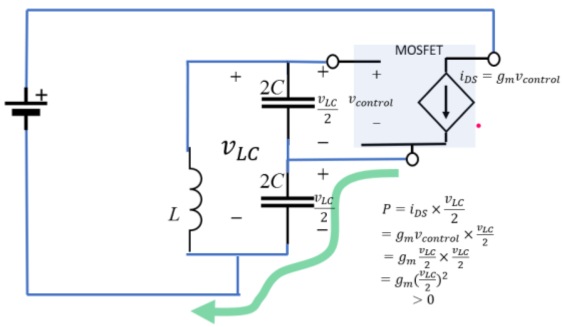
It's necessary to create more terminals for different connections. Recall we could decompose C into two 2C in series.



Then it becomes straightforward to move the transistor in following way.



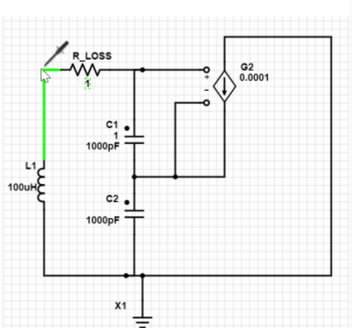
To make it neat together with current flow plotted:

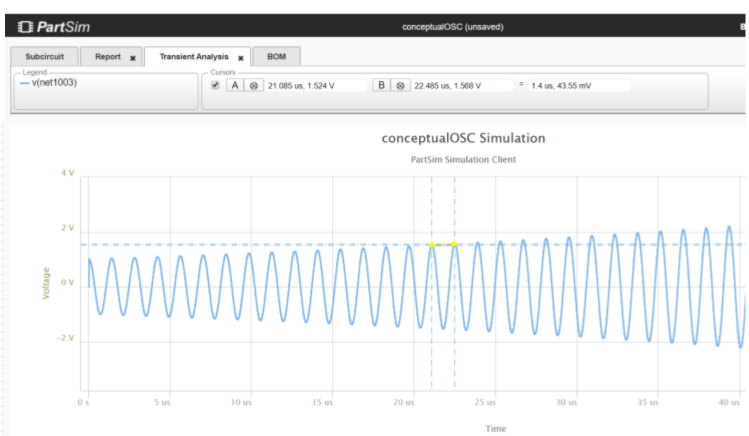


The current flows (green) through *C2*. The power delivered into the *"LC"* tank, $P = i_{DS} \times \frac{v_{LC}}{2} = g_m \frac{v_{LC}}{2} \times \left(\frac{v_{LC}}{2}\right) = g_m (\frac{v_{LC}}{2})^2 > 0$. Positive power is continuously injected into the tank to keep it energetic. I anticipate the swing will get larger and larger to build up an oscillation.

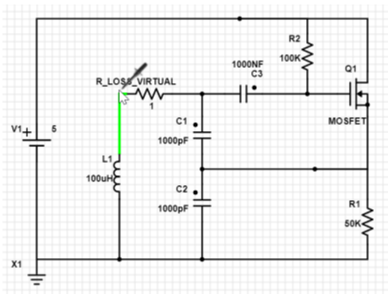
Simulation of Oscillation

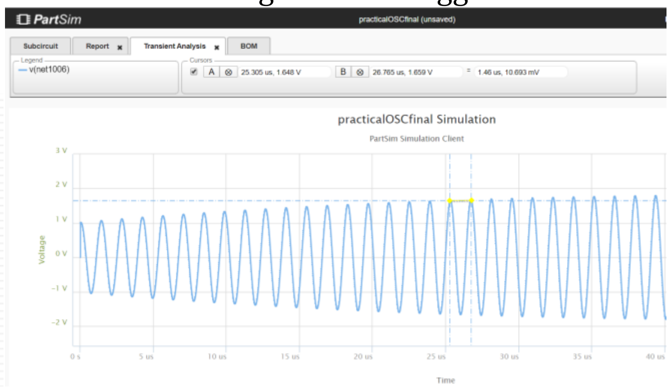
The simulation result shows my conceptual circuit is on the right track to grow oscillation.



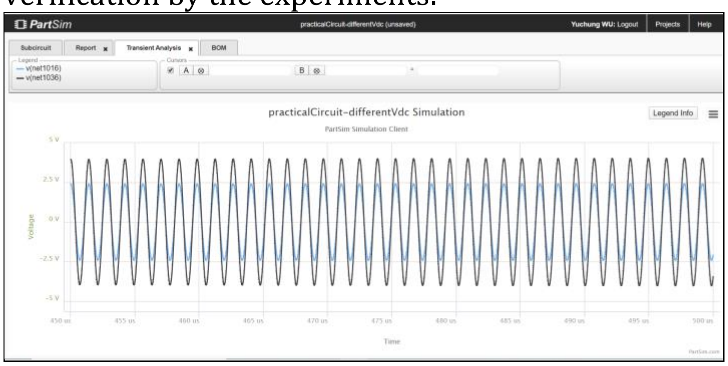


The simulation result on full circuit using MOSFET suggests it would work.





I adjusted the supply voltage from 5V to 10V. It looks like the periods are kept the same despite the oscillation amplitude grows. It's a thrilling result preliminarily and needs verification by the experiments.



Implementation 3: Experiment

References

1.Abattouy, M. (2017). The mathematics of isochronism in Galileo: From his manuscript notes on motion to the Discorsi. *ResearchGate*.

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4.Smith, S. &. (2012). *Advanced Energy Technology Group Center for Energy Research.* Retrieved from University of California San Diego: http://aries.ucsd.edu/NAJMABADI/CLASS/ECE65/12-W/Slides/ECE65_W12-MOS.pdf